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Persistence exponents and scaling in two-dimensional *XY* model and a nematic model

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Abstract
The persistence exponents associated with the $T = 0$ quenching dynamics of the two-dimensional XY model and a two-dimensional uniaxial spin nematic model have been evaluated using a numerical simulation. The site persistence or the probability that the sign of a local spin component does not change starting from initial time $t = 0$ up to a certain time t, is found to decay as $L(t)^{-\theta}(L(t)$ is the linear domain length scale), with $\theta = 0.305(\pm 0.020)$ for the two-dimensional XY model and $0.199(\pm 0.009)$ for the two-dimensional uniaxial spin nematic model. We have also investigated the scaling (at the late time of phase ordering) associated with the correlated persistent sites in both models. The persistence correlation length was found to grow in the same way as $L(t)$.

PACS numbers: 05.50.+q, 05.40.−a, 05.70.Ln, 64.70.Md

1. Introduction

Phase ordering of various systems with scalar, vector and more complex order parameters has been an active field of research over the last few years [1, 2]. When a system is suddenly quenched from a high-temperature homogeneous equilibrium phase into an ordered phase (at temperature less than the critical temperature, T_c), the system does not get ordered suddenly. Instead domains of various degenerate phases grow and in the thermodynamic limit the system develops a length scale that grows with time without any upper bound. Recently we have studied the coarsening dynamics of the two-dimensional quenched uniaxial nematic [3], where it has been established using a cell dynamic scheme [4], that in a zero temperature quenched two-dimensional nematic lattice model, dynamical scaling is obeyed and the growth law associated with the linear length scale of domains $(L(t))$ is similar to that in the two-dimensional XY model [5]. In both the systems, asymptotically, the domain length scale $L(t)$ was found to grow as $(t/\ln t)^{1/2}$. Although the interaction Hamiltonians have different symmetry, the similar structure of the topological defects supported by these models [3, 6] (both models possess stable point topological singularity) is responsible for similar asymptotic growth law of $L(t)$. So from the point of view of the growth law associated with the dynamical domain length scale $L(t)$, the coarsening dynamics are indistinguishable. However when we look for more detailed correlation that exists within the dynamically evolving non-equilibrium system, it may be possible that the two models will show different features. One such physical quantity, which probes the details of the history of the dynamics, is the persistence probability or simply the persistence. Persistence is an interesting property from both theoretical and experimental points of view in the field of non-equilibrium statistical mechanics [7].

Persistence in a general non-equilibrium process may be defined as the probability that any zero-mean stochastic variable $X(t)$ does not change sign up to certain time t starting from an initial time $t = 0$. Study of persistence in various non-equilibrium systems is of recent interest [8, 9]. Here one studies the time evolution of the order parameter field, $\phi(x, t)$, which varies in space as well as in time. Persistence in a general extended non-equilibrium system may be defined as the probability that some local order parameter (fixed at a particular point x in space) has not changed sign up to a certain time t starting from the initial time $t = 0$. This is more properly called the local or site persistence (another quantity, which is of relevant interest in the study of non-equilibrium systems, is global persistence [10, 11], which is defined in the same way for the total value of the order parameter). It is of relevant interest to see how persistence probability decays with time. The decay of persistence with time is known exactly only for Markovian processes where the two time Gaussian stationary correlator, $C(|\tau_1 - \tau_2|)$, is of pure exponential form [12–14]. However, no general answer is known for non-Markovian processes (history dependent), where the two time Gaussian stationary correlator deviates from the pure exponential form, and the decay of persistence which sensitively depends on the full form of the correlator (not just its asymptotic time form and hence becomes history dependent) becomes non-trivial. For the case of a simple one-dimensional random walk problem, which is a Markovian one, the persistence probability is found to decay as $p(t) \sim t^{-\theta}$ with θ exactly equal to $1/2$ [8, 15]. However most of the non-equilibrium dynamical processes are non-Markovian and hence calculation of the persistence exponents becomes analytically difficult. For example, the decay of persistence in the case of a simple scalar diffusion equation has been shown to be non-trivial [16, 17]. In a general non-equilibrium dynamics, the normalized two time correlator in asymptotic limit of time may be written in the simple scaling form $f(L(t)/L(t'))$ (with $t < t'$ and with the assumed validity of dynamical scaling) [1], where $L(t)$ is the diverging dynamic length scale associated with the domains in a coarsening system. The form of the two time correlator is direct outcome of the scaling hypothesis, which indicates the presence of a single characteristic length scale associated with the dynamically evolving system in late time dynamics. Clearly this process is non-stationary in real time. However if one makes the transformation $u = \ln L(t)$ [18], then the evolution of the normalized stochastic process $(X(t)/\sqrt{\langle X(t)^2\rangle})$ becomes stationary in the logarithmic scale u and the persistence probability for a Gaussian process in the asymptotic limit decays as $\mathrm{e}^{-\theta u}$ or simply as $L(t)^{-\theta}$ [12], where θ is known as the persistence exponent. In some of the papers on persistence, $P(t)$ is assumed to decay as $t^{-\theta'}$, although in general it should decay as $L(t)^{-\theta}$. This is because $L(t) \sim t^{1/z}$ is not always true, z being the dynamic growth exponent associated with the growth law of $L(t)$, (e.g. in the present systems $L(t) \sim (t/\ln t)^{1/2}$), and hence $P(t)$ is not always of the form $t^{-\theta'}$ [19]. So it will be more appropriate to designate the power of $L(t)$ in the decay as the persistence exponent. The exponent θ comes out to be independent of other dynamic exponents such as the dynamic growth exponent z and autocorrelation or Fisher–Huse exponent λ [1] (in the scaling regime the two time correlation function or the autocorrelation function is given by $C(t, t') \sim (L(t)/L(t'))^{\lambda}$, for $t' \gg t$).

Persistence of continuous spin systems, to our knowledge, has not been investigated earlier, although there is a considerable amount of numerical and analytical work on the persistence of discrete spin models such as Potts and Ising models [8, 18, 20, 21]. The experimental determination of the persistence exponent for the two-dimensional Ising model was performed by Yurke *et al* [22], using a twisted nematic film. The dependence of the persistence exponent on the updating rule of the dynamics was studied for the one-dimensional zero temperature Glauber Potts model [9, 23]. Study related to finite temperature persistence [12, 24–26] reveals that the temperature universality is not broken by this new exponent [12, 25].

In the present work we have performed a numerical simulation to obtain the persistence exponent associated with the $T = 0$ quenching dynamics of the two-dimensional spin models. These are the XY model and the uniaxial spin nematic model (where the spin dimensionality is three). As already stated, both systems obey dynamical scaling in a $T = 0$ quench and the domain length scales as $(t/\ln t)^{1/2}$ in the asymptotic limit. The purpose of the present study is twofold. It is, to our knowledge, the only work so far on the study of the persistence exponent in a continuous spin system and secondly we have investigated if the persistence exponents differ in two systems which exhibit the same asymptotic dynamical scaling growth law. We have also investigated the scaling associated with the correlated persistence sites in both models and these were found to grow in the same way as $L(t)$.

2. Simulation techniques

The Hamiltonian of a two-dimensional XY model is given by

$$H = -\sum_{\langle i,j\rangle}(\phi_i, \phi_j),$$

where ϕ is the usual two-dimensional vector spin and $\langle i, j\rangle$ represents nearest-neighbour sites. The equation of motion is given by [27],

$$\frac{\partial\phi_i}{\partial t} = \sum_j \phi_j - \sum_j (\phi_i, \phi_j)\phi_i,$$

where the sum is taken over nearest-neighbour sites. We have omitted any noise in the equation of motion, hence we are effectively working at $T = 0$.

The Hamiltonian of the two-dimensional model representing the uniaxial nematic, is given by,

$$H = -\sum_{\langle i,j\rangle}(\phi_i, \phi_j)^2,$$

where ϕ is the usual three-dimensional vector spin on a two-dimensional lattice. In this model in addition to O(3) symmetry, there exists local inversion symmetry and hence it represents a uniaxial nematic. It is also known as the spin nematic model and resembles the celebrated Lebwohl–Lasher model for uniaxial nematic, where the nearest-neighbour interaction is proportional to $-P_2(\cos\theta)$(P_2 is the second Legendre polynomial and θ is the angle between two nearest-neighbour spin vectors) [28]. Similar to the two-dimensional XY case, the equation of motion is given by [29]

$$\frac{\partial\phi_i}{\partial t} = \sum_j (\phi_i, \phi_j)\phi_j - \sum_j (\phi_i, \phi_j)^2\phi_i,$$

where the sum is taken over nearest-neighbour sites.

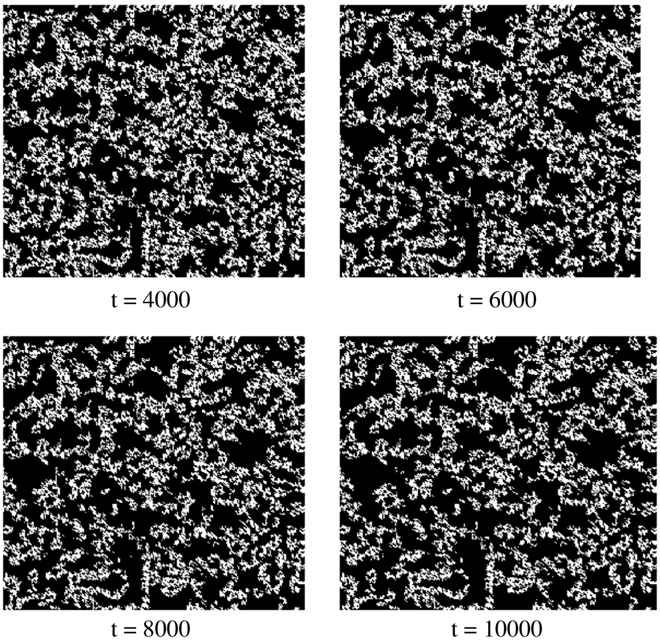

Figure 1. The persistent spins in 200 × 200 two-dimensional XY model for $t = 4000$, 6000, 8000 and 10 000 after the system is quenched from a high-temperature initial stage to $T = 0$ (white portions represent persistent sites).

We have performed a numerical simulation of discretized versions of the equations of motion. The time step δt was taken to be 0.02. However all the results shown in this paper were found to be independent of δt (for $\delta t < 0.1$) in the asymptotic regime. We have presented here results for a 400 × 400 lattice. We did not observe any significant finite size effect by comparing the results obtained for smaller lattice sizes.

3. Persistence probability and scaling of persistence correlation

The persistence probability $P(t)$ for a continuous spin system may be defined as the probability that starting from the initial time $t = 0$, any one of the components of the continuous spin at a fixed position in the lattice does not change its sign up to time t. Owing to the symmetry, one must average it over all components. We have taken the average over several random initial configurations as well as over all lattice sites. Mathematically we can write the persistence probability as

$$P(t) = \text{Probability}[S_i(t') \times S_i(0) > 0,\ \forall\, t' \text{ in } [0, t]],$$

where S_i is the ith component of the spin vector at a particular lattice site.

Scaling and fractal formation of the correlated persistence sites have attracted recent interest by various researchers [30–32]. In the present work we have investigated scaling in the spatial correlation of the persistence sites. For this we have evaluated the normalized two-point corrector,

$$C(r, t) = \langle n_i(t) n_{i+r}(t)\rangle / \langle n_i(t)\rangle$$

where $\langle\ \rangle$ represents the average over sites as well as random initial conditions. $n_i(t) = 1$ if the ith site is persistent, otherwise it is 0. This correlation just gives the probability that

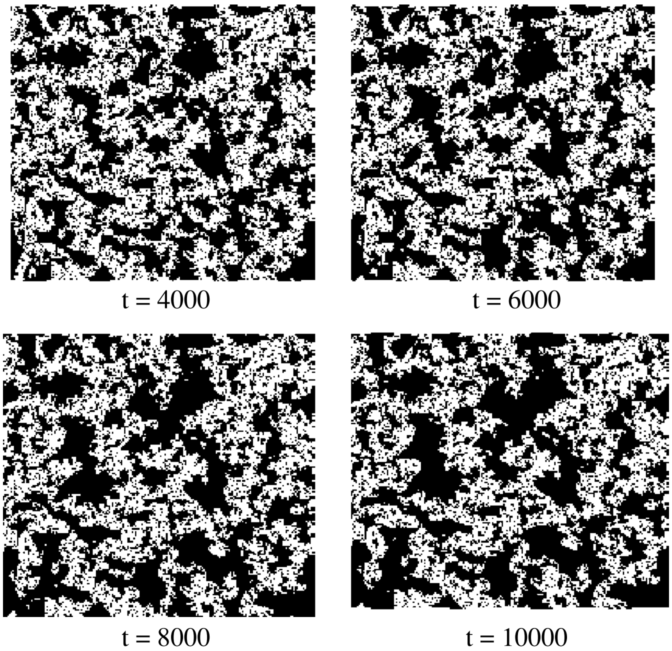

Figure 2. The persistent spins in 200 × 200 two-dimensional spin nematic model for $t = 4000$, 6000, 8000 and 10 000 after the system is quenched from a high-temperature initial stage to $T = 0$ (white portions represent persistent sites).

the spin at $(i + r)$th site is persistent, given the ith site is persistent. Beyond a certain length $\xi(t)$ (persistence correlation length), the sites are found to be uncorrelated, and $C(r, t)$ is simply $\langle n(t)\rangle$ or the persistence probability $P(t)$. However for $r < \xi(t)$ there exists strong correlation. In the correlated region $C(r, t)$ shows a power-law decay with distance $r^{-\alpha}$ and hence is independent of t or $L(t)$. So for $r < \xi(t)$, there exists strong correlation with scale invariant behaviour, which indicates the expected self-similar fractal structure formed by the persistent sites [9, 30, 31]. Now at $r = \xi(t)$, consistency demands $\xi^{-\alpha}(t) \sim L(t)^{-\theta}$ (since $P(t) \sim L(t)^{-\theta}$), which simply implies $\xi(t)$ should diverge as $L(t)^{\zeta}$ with $\zeta = \theta/\alpha$. Mathematically we can write $C(r, t)$ as

$$\begin{aligned} C(r,t) &\sim r^{-\alpha} && \text{for} \quad r \ll \xi(t) \\ &= P(t) && \text{for} \quad r \gg \xi(t). \end{aligned}$$

Clearly in scaling form $C(r, t)$ can be written as

$$C(r,t) = P(t) f(r/\xi(t)),$$

where $f(x)$ is given by

$$\begin{aligned} f(x) &\sim x^{-\alpha} && \text{for} \quad x \ll 1 \\ &= 1 && \text{for} \quad x \gg 1. \end{aligned}$$

4. Results and discussions

In figures 1 and 2 we have shown how correlated regions of persistence sites are formed in the two-dimensional XY and the two-dimensional spin nematic models at various times t, after

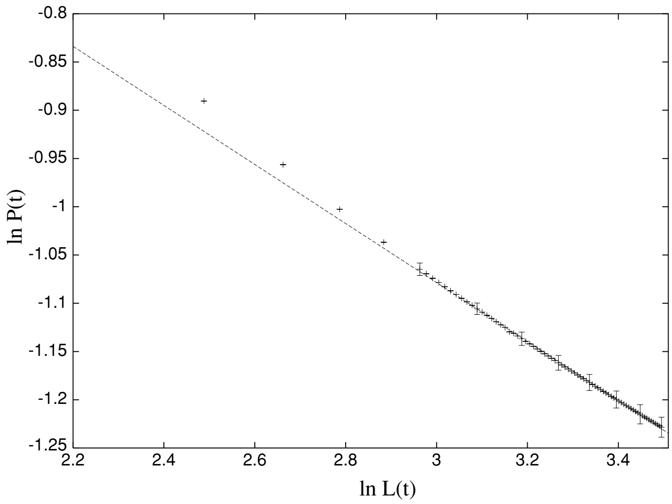

Figure 3. Plot of ln $P(t)$ against ln $(L(t))$ for 400×400 XY model. The linearity of the plot in the asymptotic time limit ensures the decay of the form $P(t) = L(t)^{-\theta}$ or $(t/\ln t)^{-\theta/2}$, with $\theta = 0.305$ (± 0.020). The linear region extends from $t = 3000$ to $t = 10\,000$. Average over 12 initial configurations and 400×400 sites were taken.

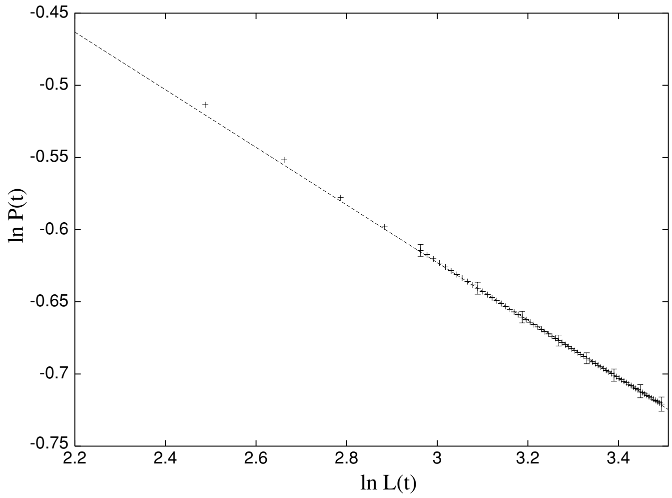

Figure 4. Plot of ln $P(t)$ against ln $L(t)$ for 400×400 spin nematic model. The linearity of the plot in the asymptotic limit ensures the decay $P(t) = L(t)^{-\theta}$ or $(t/\ln t)^{-\theta/2}$, with $\theta = 0.199$ (± 0.009). The linear region extends from $t = 3000$ to $t = 10\,000$. Average over 15 initial configurations and 400×400 sites were taken.

the system was quenched from the initial homogeneous $T = \infty$ configuration. In figure 3 we have shown the decay of the persistence with $L(t) = (t/\ln t)^{1/2}$ for the two-dimensional XY model. The linearity in the log–log plot reflects the decay to be of the form $P(t) = L(t)^{-\theta}$ or $(t/\ln t)^{-\theta/2}$ in the late time regime. The exponent θ we obtained was 0.305 (± 0.020). In figure 4 we have depicted the same for the two-dimensional spin nematic model and the exponent θ we obtained was 0.199 (± 0.009). The error bars provided are calculated by

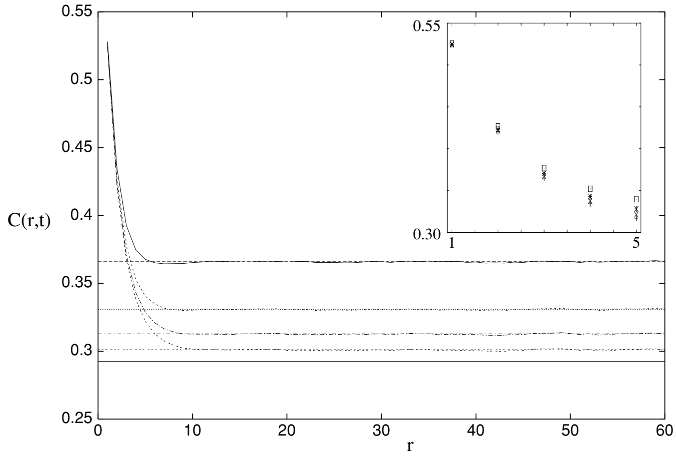

Figure 5. The variation of correlation function with distance for the 400 × 400 two-dimensional XY model. (a) In the main figure it is shown that, for large r, $C(r, t)$ is the same as persistence probability (lines parallel to the x-axis represents $P(t)$). The data are for time steps $t = 2000$, 4000, 6000, 8000 and 10 000 (from top to bottom) with persistence probability $P(t) = 0.366$, 0.331, 0.313, 0.301 and 0.292 respectively. (b) Inset shows $C(r, t)$ for $t = 4000$, 6000, 8000 and 10 000, for small values of r ($\leqslant 5$). The overlapping values of $C(r, t)$ in the neighbourhood of $r = 0$ for different time steps verify that $C(r, t)$ is independent of t for small values of r.

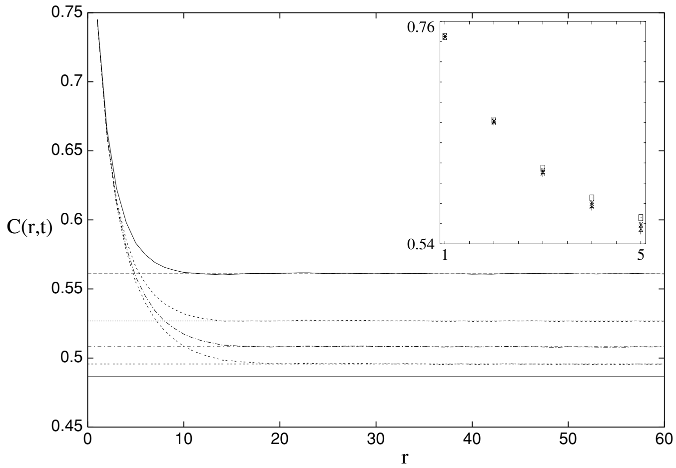

Figure 6. The variation of correlation function with distance for the 400 × 400 two-dimensional spin nematic model model. (a) In the main figure it is shown that, for large r, $C(r, t)$ is the same as persistence probability (lines parallel to the x-axis represent $P(t)$). The data are for time steps $t = 2000$, 4000, 6000, 8000 and 10 000 (from top to bottom) with persistence probability $P(t) = 0.561$, 0.527, 0.508, 0.496 and 0.487 respectively. (b) Inset shows $C(r, t)$ for $t = 4000$, 6000, 8000 and 10 000, for small values of r ($\leqslant 5$). The overlapping values of $C(r, t)$ in the neighbourhood of $r = 0$ for different time steps verify that $C(r, t)$ is independent of t for small values of r.

estimating the standard deviation of the values of θ obtained from power-law fits available from the simulations with different initial configuration.

In figures 5 and 6 we have shown the correlator $C(r, t)$ plotted against r for various values of t for the XY model and the spin nematic model. In both figures it is observed that for small

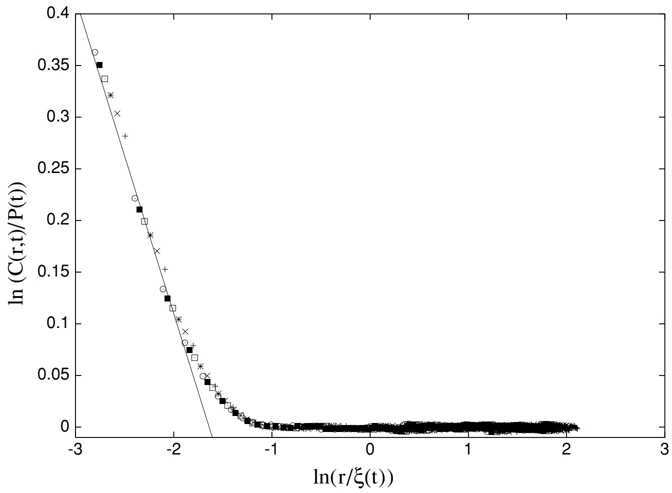

Figure 7. Plot of $\ln(C(r,t)/P(t))$ against $\ln(r/\xi(t))$. The best collapse is obtained when the value of $\zeta = 1$ (± 0.01), i.e. if $\xi(t) \sim (t/\ln t)^{1/2}$. The straight line for small values of $r/\xi(t)$ has slope α equal to 0.305, which is equal to the persistence exponent of the two-dimensional XY model. The data used are for time $t = 5000$, 6000, 7000, 8000, 9000 and 10 000.

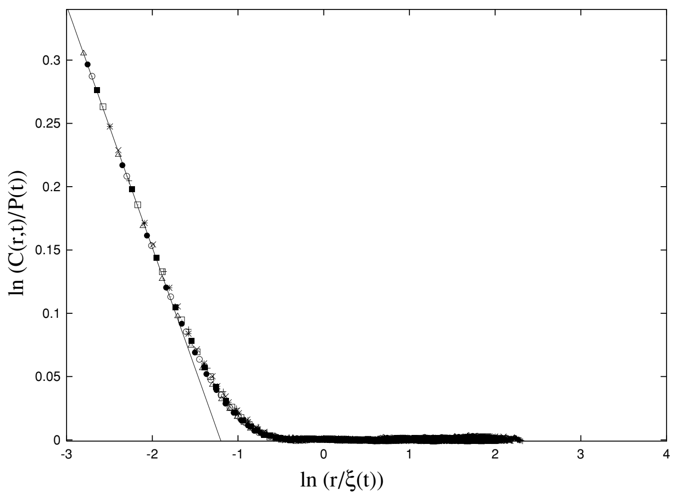

Figure 8. Plot of $\ln(C(r,t)/P(t))$ against $\ln(r/\xi(t))$. The best collapse is obtained when the value of $\zeta = 1$ (± 0.006), i.e. if $\xi(t) \sim (t/\ln t)^{1/2}$. The straight line for small values of $r/\xi(t)$ has slope α equal to 0.191, which is almost equal to the persistence exponent of the two-dimensional spin nematic model. The data used are for time $t = 3000$, 4000, 5000, 6000, 7000, 8000, 9000 and 10 000.

values of r, $C(r,t)$ for each time overlaps and for large values of r, $C(r,t)$ is equal to $P(t)$. For small values of r, a $r^{-\alpha}$ decay is observed. In figures 7 and 8 we have shown the log–log plot of the scaling function of $C(r,t)$ for the XY model and the spin nematic model. We obtained good collapse for $\zeta = 1$ (and hence $\alpha = \theta$) which implies that the persistence correlation length $\xi(t)$ diverges as $L(t)$ or $(t/\ln t)^{1/2}$. It is of interest to note that the persistence correlation length has similar divergence as that of the length scale associated with the domains formed during coarsening of the system. The error bars of ζ (mentioned in the figure captions) are quite rough here, calculated by estimating the region over which the collapse appears optimal. We point out that we have also tried to collapse our data with the familiar form of the growth law $t^{1/2}$, but were unable to obtain a good collapse.

5. Conclusion

We would like to summarize the main findings of the paper. In the present work we have studied the site persistence in the $T = 0$ quenching dynamics of the two-dimensional XY model and the two-dimensional spin nematic model. Although in both the models, the dynamical domain length scales $L(t)$ have similar growth laws in the asymptotic limit, the persistence exponents come out to be different. In the XY model, it is $0.305(\pm 0.020)$ while in the spin nematic model it is $0.199(\pm 0.009)$. We have also investigated the scaling structure of persistence sites for both the models. We obtained the growth law of persistence correlation length to be the same as that of the domain length scale $L(t)$, i.e. $(t/\ln t)^{1/2}$.

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